Advanced Quantum Mechanics

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Exercise sheet 7 (Due: Monday November, 25$^{\text{th}}$.)

7.1 Taming an infinity

In the lecture we learned that the expected electric field, $\langle \vec{E}(\vec{x}) \rangle$, at any given point $\vec{x}$ is zero for any Fock state with respect to the momentum modes (and thus also for the vacuum state), but that the strength of the fluctuations in the field, $\langle \|\vec{E}(\vec{x})\|^2 \rangle$, diverges to infinity for the vacuum state. Since then you have most likely lost sleep from fear that you suddenly might explode due to some large vacuum fluctuations. In order to put your fears to rest, we will here show that this infinity can be tamed. Recall that one can think of $\langle \vec{E}(\vec{x}) \rangle$ and $\langle \|\vec{E}(\vec{x})\|^2 \rangle$ as the effects on a point-charge. However, let us now instead consider the force

$$\vec{F} = \int \rho(\vec{x}) \vec{E}(\vec{x}) d^3x, \tag{1}$$

on some charge distribution $\rho(\vec{x})$, and where the electric field operator is[1]

$$\hat{\vec{E}}(\vec{x}) = -i\sqrt{\frac{2\hbar\pi}{L^3}} \sum_{\vec{k},\lambda} \sqrt{\omega_k} \vec{e}_\lambda(\vec{k})(a^\dagger_{\vec{k},\lambda} e^{-i\vec{k}\cdot\vec{x}} - a_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{x}}). \tag{2}$$

(a) In the lecture we showed that expectation value of $\vec{E}$ is zero for any Fock-state $|n_1, n_2, \ldots\rangle$ for all the $(\vec{k}, \lambda)$-modes. Here we wish to show the same for the force $\vec{F}$. *Hence, show that* $\langle n_1, n_2, \ldots | \vec{F} | n_1, n_2, \ldots \rangle = 0$ *for any Fock-state* $|n_1, n_2, \ldots\rangle$. **(3 points)**

Remark: As a special case, we thus get that the expectation of the force due to the vacuum state is zero.

(b) Now we turn to the expectation value of $\|\vec{F}\|^2$ with respect to the vacuum state $|\mathrm{vec}\rangle$. This measures the magnitude of the fluctuations in the vacuum field. *Show that*

$$\langle \mathrm{vac}| \, \|\vec{F}\|^2 \, |\mathrm{vac}\rangle = \frac{2\hbar\pi}{L^3} \sum_{\vec{k}} \omega_k |\tilde{\rho}(\vec{k})|^2, \tag{3}$$

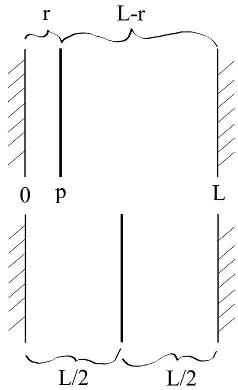
where $\tilde{\rho}(\vec{k}) = \int \rho(\vec{x}) e^{i\vec{k}\cdot\vec{x}} d^3x$. **(5 points)**

Remark: This result means that if $|\tilde{\rho}(\vec{k})|^2$ goes to zero sufficiently fast as $\|\vec{k}\| \to \infty$, then the sum (3) becomes finite. Hence, the infinity can be tamed. Note also that the more smeared out $\rho(\vec{x})$ is, the faster the Fourier transform $\tilde{\rho}(\vec{k})$ decays, and thus the smaller the effect of the fluctuations.

[1] This is the ninja-edited version.

7.2 The Casimir force

One can wonder whether the vacuum energy only is a mathematical absurdity without any physical meaning. However, one can argue that the *change* in the vacuum energy actually has physical consequences, in terms of the Casimir force (or Casimir effect). Here we shall investigate this in a simplified one-dimensional model.



Consider one-dimensional "cavity" of length L, where we insert a plate p that divides the cavity into two parts. We thus get two cavities, one of length r and one of length $L - r$. Now consider the smaller cavity of length r. Standing waves in that cavity can be formed at frequencies $\omega_k = \frac{k\pi}{r}$ for $k = 1, \ldots$. On these modes we have the annihilation operators $a_{k,\lambda}$ and creation operators $a^\dagger_{k,\lambda}$ (where λ is the polarization), and we get the Hamiltonian $H = \sum_{k,\lambda} \hbar\omega_k (a^\dagger_{k,\lambda} a_{k,\lambda} + \frac{1}{2}\hat{1})$. For the vacuum state $|\text{vac}\rangle$, the expectation value of the energy is $E(r) = \langle \text{vac}|H|\text{vac}\rangle = \hbar \sum_k \omega_k = \hbar \sum_k \frac{\pi k}{r}$. For the combined cavities r and $L - r$, the vacuum energy thus is $E(r) + E(L - r)$, which we know is infinite. Recall that absolute energies often do not have much meaning, but that energy differences do. So, let us compare with the case where we insert the wall in the middle of the cavity.[2] The difference in energy is $\Delta E(r) = E(r) + E(L - r) - 2E(L/2)$. However, we do not really gain much, since we here have an expression of the form $\infty + \infty - 2\infty$. We will now regularize these infinities by introducing a cutoff-function $g(\omega) = e^{-\xi\omega}$, where the regularized energy function is[3]

$$E(r, \xi) = \hbar \sum_{k=1}^{\infty} \omega_k g(\omega_k) = \hbar \sum_{k=1}^{\infty} \frac{\pi k}{r} e^{-\xi \frac{\pi k}{r}}. \tag{4}$$

The function $E(r, \xi)$ is finite for all $\xi > 0$, and $E(r, 0) = E(r)$. The role of the cutoff-function is to gradually cut off the high frequencies. As ξ goes to zero, the higher up in frequency we move the cutoff.

(a) A good thing about this particular cutoff function is that we can find a closed expression for the regularized energy. *Show that*

$$E(r, \xi) = \frac{\hbar\pi}{r} \frac{e^{-\xi \frac{\pi}{r}}}{(1 - e^{-\xi \frac{\pi}{r}})^2}. \tag{5}$$

(3 points)

(b) We are interested in what happens for small values of ξ (i.e., when the cutoff only significantly affects very high frequencies).

Show that

$$E(r, \xi) = \alpha \frac{r}{\xi^2} + \beta \frac{1}{r} + O(\xi) \tag{6}$$

and determine the constants α and β. **(5 points)**

[2] There is nothing special with putting the wall in the middle. Any fixed reference position would do.

[3] Do you think that this regularization business is cheating? Well, welcome to the world of quantum field theory!

(c) We define the regularized energy difference

$$\Delta E(r,\xi) = E(r,\xi) + E(L-r,\xi) - 2E(L/2,\xi). \tag{7}$$

We now wish to determine this energy difference as a function of $r > 0$ in the limit where we take $\xi \to 0$ (thus take the cutoff to infinity), and where we make the larger cavity infinitely large $L \to \infty$ (so that we get the effective force between p and wall 0). *Show that*

$$\lim_{L\to\infty} \lim_{\xi\to 0} \Delta E(r,\xi) = -\frac{\gamma}{r}, \tag{8}$$

and determine the constant $\gamma > 0$. *What is thus the effective force between the plate p and the wall* 0, *as a function of of the distance* r*? Is the force attractive or repulsive between the wall and the plate?* **(4 points)**

Remark: With a more accurate model, where one treats the walls and the plate as extended surfaces, one finds that the force is proportional to r^{-4}. The existence of the Casimir effect has been confirmed by experiments.