Advanced Quantum Mechanics

D. Gross, J. Åberg

Exercise sheet 8 (Due: Monday December, 2nd.)

8 Coherent states of oscillators and fields

For quantum harmonic oscillators, the evolution of coherent states shadows that of the corresponding classical oscillator. Since the free EM field can be regarded as an infinite collection of harmonic oscillators, it is maybe not surprising that there is a similar correspondence. Here we shall explore this. Apart from the physical relevance, coherent states are also very useful, and pop up in all kinds of places, so it is very good to know about their properties.

For a single mode, the family of coherent states can be written

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{1}$$

for any $\alpha \in \mathbb{C}$.

(a) *What is the overlap $\langle\alpha|\beta\rangle$? Express $|\langle\alpha|\beta\rangle|^2$ in terms of $|\alpha-\beta|^2$. What does this say about the approximate orthogonality of $|\alpha\rangle$ and $|\beta\rangle$ as $|\alpha-\beta|^2$ grows?* **(2 points)**

(b) In spite of the fact that the coherent states are not orthogonal to each other (and form an overcomplete set), they nevertheless satisfy a resolution of identity. *Show that*

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha = \hat{1}, \tag{2}$$

where $d^2\alpha = d\mathrm{Re}(\alpha)d\mathrm{Im}(\alpha)$.

Hint: It might be useful to know the value of the integral $\int_0^\infty e^{-s}s^n ds$. **(3 points)**

(c) *Show that*

$$|\psi(t)\rangle = e^{-it\omega/2}|\alpha(t)\rangle, \quad \alpha(t) = \alpha e^{-it\omega} \tag{3}$$

is the solution to the time dependent Schrödinger equation with respect to the Hamiltonian $H = \hbar\omega(a^\dagger a + \frac{1}{2}\hat{1})$ of the harmonic oscillator, and the initial condition $|\psi(0)\rangle = |\alpha\rangle$. **(3 points)**

Remark: By identifying the complex plane with phase space, we can interpret $\alpha(t) = \alpha e^{-it\omega}$ as the evolution of the position and momentum of the classical oscillator.

(d) Recall that the quantum vector potential is $\hat{\vec{A}}(\vec{x}) = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\vec{k},\lambda} \frac{\vec{e}_\lambda(\vec{k})}{\sqrt{\omega_k}} (a^\dagger_{\vec{k},\lambda} e^{-i\vec{k}\cdot\vec{x}} + a_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{x}})$. Now consider the quantum EM field, where we assume that each $\vec{k}, \lambda$-mode is in a coherent state $|\alpha_{\vec{k},\lambda}(t)\rangle$, where $\alpha_{\vec{k},\lambda}(t) = \alpha_{\vec{k},\lambda} e^{-it\omega_k}$. Let

$$|\chi(t)\rangle = \bigotimes_{\vec{k},\lambda} e^{-\frac{1}{2}it\omega_k} |\alpha_{\vec{k},\lambda}(t)\rangle, \quad \alpha_{\vec{k},\lambda}(t) = \alpha_{\vec{k},\lambda} e^{-it\omega_k}. \tag{4}$$

Determine the expectation value $\vec{A}(\vec{x},t) = \langle \chi(t) | \hat{\vec{A}}(\vec{x}) | \chi(t) \rangle$. This expectation value is nothing but a classical vector potential of a free field. *Confirm this by showing that* $\vec{A}(\vec{x},t)$ *satisfies the Maxwell equation of the free-field vector potential,* $(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\vec{A}(\vec{x},t) = 0$, *and that it satisfies the Couloumb gauge condition* $\nabla \cdot \vec{A}(\vec{x},t) = 0$.

Hint: Recall that the polarization vectors $\vec{e}_\lambda(\vec{k})$ are orthogonal to the direction of propagation $\vec{k}$, i.e., $\vec{e}_\lambda(\vec{k}) \cdot \vec{k} = 0$. **(4 points)**

(e) *Show that* $|\chi(t)\rangle$ *in (4) satisfies the time-dependent Schrödinger equation for the field Hamiltonian* $H = \sum_{\vec{k},\lambda} \hbar\omega_k (a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{1}{2}\hat{1})$.

Hint: Keep in mind exercise (c). **(3 points)**

Remark: Exercise (d) shows that the expectation value of the quantum vector potential with respect to the coherent field state follows the evolution of the classical counterpart. Conversely, by exercise (e) we know that the classical field gives the evolution of the corresponding field coherent state. Does this mean that field coherent states precisely correspond to classical fields? No, not quite, and that has to do with the intrinsic uncertainties in the quantum case. For the harmonic oscillator, the coherent states are minimal uncertainty states. The field coherent states have similar properties.

(f) Analogous to what we did in exercise 7.1, let us investigate the fluctuations of the force $\vec{F} = \int \rho(\vec{x}) \hat{\vec{E}}(\vec{x}) d^3x$, on a charge distribution $\rho(\vec{x})$, for the electric field operator

$$\hat{\vec{E}}(\vec{x}) = -i\sqrt{\frac{2\hbar\pi}{L^3}} \sum_{\vec{k},\lambda} \sqrt{\omega_k} \vec{e}_\lambda(\vec{k}) (a^\dagger_{\vec{k},\lambda} e^{-i\vec{k}\cdot\vec{x}} - a_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{x}}), \tag{5}$$

with respect to the coherent field state $|\chi(t)\rangle$ as in (4).

Show that

$$\Delta F(t) := \langle \chi(t) | \, \|\vec{F}\|^2 \, | \chi(t) \rangle - \| \langle \chi(t) | \vec{F} | \chi(t) \rangle \|^2 = \frac{2\hbar\pi}{L^3} \sum_{\vec{k},\lambda} \omega_k |\tilde{\rho}(\vec{k})|^2. \tag{6}$$

with $\tilde{\rho}(\vec{k}) = \int \rho(\vec{x}) e^{i\vec{k}\cdot\vec{x}} d^3x$.

Hint: There is a massive cancellation of terms between $\langle \chi(t) | \, \|\vec{F}\|^2 \, | \chi(t) \rangle$ and $\| \langle \chi(t) | \vec{F} | \chi(t) \rangle \|^2$. **(5 points)**

Remark: We can conclude that the fluctuations will be non-vanishing in general. Note also that these fluctuations are independent of the choice of coherent field state, and that they are the same as those that we obtained for the vacuum state (see 7.1). One should keep in mind that the vacuum state actually is a coherent state.

Think about this: The field coherent state $|\chi(t)\rangle$ is determined by the collection of complex numbers $\alpha_{\vec{k},\lambda}$. Suppose that we would use the very same numbers to determine a classical electric field. What would the variance of the force be?[1]

[1]If one consider an alternative where $\alpha_{\vec{k},\lambda}$ are random variables, then one can get a nonzero variance of the

force also in the classical case. This corresponds to models of noisy fields, e.g., due to thermal fluctuations.