Advanced Quantum Mechanics

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Exercise sheet 9 (Due: Monday December, 9th.)

9.1 Lineshape of absoption spectra

In the lecture we discussed decay, Fermi's golden rule, and the associated spectral line shape. In absorption spectroscopy, one records the strength of the absorption as a function of how far the frequency of the light source is from the transition frequency. Here we use the Jaynes-Cummings model to analyze this. However, in the lecture we only discussed the JC-model "on resonance", while we here need to allow for "detuning", which means that the transition frequency of the atom ω_0 is different from the transition frequency ω of the mode. This more general JC-model reads

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g\sigma_+ a + \hbar g\sigma_- a^\dagger. \tag{1}$$

We let the detuning be denoted by $\delta = \omega - \omega_0$.

(a) In the lecture, we showed that the Jaynes-Cummings Hamiltonian on resonance decomposes into a collection of block diagonal matrices in the $|\pm, n\rangle$ basis. *Show that the same decomposition also holds for the more general model (1) and determine the corresponding block-matrices.*

(2 points)

(b) *Show or confirm that the eigenvalues and normalized eigenvectors of H are*

$$\begin{aligned}
E_0 =& -\frac{\hbar\omega_0}{2}, \quad |\psi_0\rangle = |-,0\rangle, \\
E_{\pm,n} =& \hbar\omega(n-\frac{1}{2}) \pm \hbar\Omega, \qquad \Omega = \sqrt{\frac{\delta^2}{4} + g^2 n}, \\
|\psi_{\pm,n}\rangle =& \frac{1}{\sqrt{2\Omega}}\left(\mp\sqrt{\Omega \mp \frac{\delta}{2}}|+,n-1\rangle - \sqrt{\Omega \pm \frac{\delta}{2}}|-,n\rangle\right), \quad n = 1, 2, \ldots
\end{aligned} \tag{2}$$

(3 points)

(c) *Show that the family of evolution operators with respect to H is given by*

$$\begin{aligned}
U(t) =& e^{\frac{it\omega_0}{2}}|-,0\rangle\langle -,0| \\
&+ e^{-it\omega(n-\frac{1}{2})}\Bigg(\left[\cos(\Omega t) - i\frac{\delta}{2\Omega}\sin(\Omega t)\right]|-,n\rangle\langle -,n| \\
&\qquad + \left[\cos(\Omega t) + i\frac{\delta}{2\Omega}\sin(\Omega t)\right]|+,n-1\rangle\langle +,n-1| \\
&\qquad - i\frac{g\sqrt{n}}{\Omega}\sin(\Omega t)\Big[|+,n-1\rangle\langle -,n| \; + \; |-,n\rangle\langle +,n-1|\Big]\Bigg).
\end{aligned} \tag{3}$$

(3 points)

(d) We construct our model of absorption spectroscopy by assuming that the atom is in the ground state, and that the mode contains one single photon. *Determine the probability $P_+(\delta, t)$ that the the atom absorbs the photon. Give the answer in terms of g, δ, and t.*

As opposed to ordinary spectroscopy, where the photon is irreversibly absorbed (or re-emitted in other directions), we here have an oscillatory transition between the atom and the mode. To deal with this, we take as the "line shape", the function $f(\delta) = \max_t P_+(\delta, t)$ (i.e., $f(\delta)$ is the envelope of the oscillating function). *Determine the line shape.* The width of the line shape determines how fuzzy the spectral line is. *Qualitatively, what happens with the width as the coupling g increases?* As one can see from (3), the frequency Ω determines how fast the photon is absorbed (and re-emitted). *At $\delta = 0$, how does Ω depend on g?*

(2 points)

9.2 Squeezed states

In quantum optics one can implement various operations, and produce all kinds of strange quantum states. One such operation is the squeezing operator. On a mode with bosonic annihilation and creation operator a and $a^\dagger$, the squeezing operators are defined as $S(z) = e^{\frac{1}{2}(z^* a^2 - z(a^\dagger)^2)}$, $z \in \mathbb{C}$.

(a) *Show that $S(z)$ is unitary.* **(1 point)**

(b) Let $z = re^{i\theta}$ with $r \geq 0$. *Show that*

$$\begin{aligned} S(z)^\dagger a^\dagger S(z) &= \cosh(r) a^\dagger - e^{-i\theta} \sinh(r) a, \\ S(z)^\dagger a S(z) &= \cosh(r) a - e^{i\theta} \sinh(r) a^\dagger. \end{aligned} \tag{4}$$

Hint: Use the Baker-Campbell-Hausdorff theorem, which states that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots . \tag{5}$$

Induction can be useful in order to evaluate the nested commutators. **(4 points)**

(c) The operation $S(z)$ (but not expressed in this manner) has figured in previous exercises, under a different name.[1] *Which name did we associate to this transformation?* We also derived a condition for that type of transformation. *Show that the mapping in (4) satisfies that condition. What can we conclude about the commutation relations of the operators $S(z)^\dagger a S(z)$ and $S(z)^\dagger a^\dagger S(z)$ from this observation?* **(2 points)**

(d) In the lab one can implement the squeezing operator, and it can for example be applied to the vacuum state, in which case one gets a "squeezed vacuum". The squeezed vacuum corresponds to the state vector $|\psi_z\rangle = S(z)|0\rangle$. To the mode we can associate[2]

$$q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger), \tag{6}$$

and for a state $|\psi\rangle$ we can determine the corresponding uncertainties

$$\Delta q_{|\psi\rangle} = \sqrt{\langle\psi|q^2|\psi\rangle - \langle\psi|q|\psi\rangle^2}, \quad \Delta p_{|\psi\rangle} = \sqrt{\langle\psi|p^2|\psi\rangle - \langle\psi|p|\psi\rangle^2} \tag{7}$$

Determine $\Delta q_{|\psi_z\rangle}$ and $\Delta p_{|\psi_z\rangle}$. For the case that z is real, compare $\Delta q_{|\psi_z\rangle}$ and $\Delta p_{|\psi_z\rangle}$ as a function of r. In the same case, determine the product $\Delta q_{|\psi_z\rangle}\Delta p_{|\psi_z\rangle}$. Relate to the uncertainty relation. Maybe you can see why $S(z)$ is called the "squeezing operator".

(3 points)

[1] It is not unusual that the same concept is invented independently in different branches of physics and mathematics. It can take quite a while before anyone notices it.

[2] These are of course not the position and momentum operators. They are often referred to as the "quadrature operators".