Test exam for advanced Quantum Mechanics

1. Traces, partial traces and coherent states

(a) Let $|q\rangle$ and $|r\rangle$ be two arbitrary state-vectors on some Hilbert space. *Show that*

$$\mathrm{Tr}(|q\rangle\langle r|) = \langle r|q\rangle. \tag{1}$$

(b) Coherent states can be written

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2}$$

where $\alpha \in \mathbb{C}$. For two coherent states $|\alpha\rangle, |\beta\rangle$, determine the overlap $\langle\alpha|\beta\rangle$. Express the answer in terms of the complex numbers α and β.

(c) On two bosonic modes, we have a joint state described by the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle_1 \otimes |0\rangle_2 + \frac{1}{\sqrt{2}}|\beta\rangle_1 \otimes |1\rangle_2, \tag{3}$$

where $|\alpha\rangle_1, |\beta\rangle_1$ are coherent states on mode 1, and and where $|0\rangle_2, |1\rangle_2$ is the vacuum and single-particle state on mode 2.

Determine the reduced density operator on mode 2. *Express the answer in terms of the state vectors* $|0\rangle_2, |1\rangle_2$, *and the complex numbers* α *and* β.

2. Identical particles

(a) Let $|2, 0, 1\rangle$ be a bosonic Fock state with respect to the orthonormal single particle states $\phi_\alpha, \phi_\beta, \phi_\gamma$ (in that order). Write down the corresponding three-particle wave-function $\psi(x_1, x_2, x_3) = \langle x_1, x_2, x_3|2, 0, 1\rangle$ in terms of the single-particle wave-functions $\phi_\alpha(x)$, $\phi_\beta(x)$, and $\phi_\gamma(x)$.

(b) Suppose that a single particle has the energy spectrum $E_n = \hbar\omega(n + \frac{1}{2})$, for $n = 0, 1, 2, \ldots$.

- Suppose that we have 4 identical, non-interacting bosons of this type, what would the ground state energy of these four bosons be?
- What would the ground state energy be if we instead have 4 identical non-interacting spin-$\frac{1}{2}$ fermions?

3. Conservation of particle-number for one-body Hamiltonians: Fermionic case

Suppose that we have a Hamilton operator of the form $H = \sum_{jk} T_{jk} a_j^\dagger a_k$, where a_j and $a_j^\dagger$ are fermionic annihilation and creation operators, and T_{jk} are complex numbers. Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H.

Hint: Recall the relations $[AB, C] = A[B, C] + [A, C]B$, $[A, BC] = B[A, C] + [A, B]C$, and $[AB, C] = A\{B, C\} - \{A, C\}B$, where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ is the anti-commutator.

4. Jaynes-Cummings model

Suppose that a single mode (e.g., in a cavity) of the electromagnetic field interacts with a two-level system via the (on resonance) Jaynes-Cummings Hamiltonian

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega_0 a^\dagger a + \hbar g \sigma_+ a + \hbar g \sigma_- a^\dagger, \tag{4}$$

where $a, a^\dagger$ are the annihilation and creation operator of the mode, and $\sigma_z = |+\rangle\langle+| - |-\rangle\langle-|$, $\sigma_+ = |+\rangle\langle-|$ and $\sigma_- = |-\rangle\langle+|$, where $|+\rangle, |-\rangle$ are orthonormal.

- *Show that the Jaynes-Cummings Hamiltonian (4) in the $|\pm, n\rangle$ basis decomposes into a collection of block diagonal matrices.*
- *Use this to find the eigenvalues and corresponding normalized eigenvectors of (4).*
- Suppose that the mode starts in a state with 100 excitations, and the two-level system in state $|-\rangle$. *Under the evolution by Hamiltonian (4), will there ever (i.e., for any evolution time) be a non-zero probability to find* 80 *excitations in the cavity?* (Tell how you reach your conclusion.)

Hint: Recall that $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$.

5. Dirac equation

The Dirac equation can be written on the form

$$i\frac{\partial}{\partial t}\Psi = -i\sum_{l=1}^{3}\alpha^l\partial_l\Psi + m\beta\Psi, \tag{5}$$

where Ψ is a spinor, and where we have put $\hbar = 1$ and $c = 1$.

Make an ansatz in the form of plane waves in (5)

$$\Psi_{\vec{p}}(t,\vec{r}) = we^{-i[E_{\vec{p}}t-\vec{p}\cdot\vec{r}]}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}. \tag{6}$$

By inserting $\Psi_{\vec{p}}$ into (5), and using the representation where

$$\alpha^k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}, \tag{7}$$

the Dirac equation (5) reduces to an eigenvalue problem for the vectors w, i.e, $Mw = E_{\vec{p}}w$.

- *Determine the matrix M.* (It is enough to express this matrix in terms of the Pauli matrices σ_k and the identity matrix $\mathbb{I}$.)
- *Determine an orthonormal set of eigenvectors of M for particles at rest, i.e., plane-wave solutions with momentum $\vec{p} = 0$. Identify the positive and negative energy solutions.*

6. Scattering on a spherically symmetric potential

The first order Born approximation to the scattering amplitude is given by

$$f^{(1)}(\vec{k}_{\text{out}},\vec{k}_{\text{in}}) = -\frac{m}{2\pi\hbar^2}\int V(\vec{r})e^{-i(\vec{k}_{\text{out}}-\vec{k}_{\text{in}})\cdot\vec{r}}d^3r, \tag{8}$$

where $\vec{k}_{\text{in}}$ and $\vec{k}_{\text{out}}$ are the incoming and outgoing wave-vectors, V is the potential, and m the mass of the particle.

(a) *In the case that the potential is rotationally symmetric, show that (8) simplifies to*

$$f^{(1)}(q) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin(qr)dr, \tag{9}$$

where $q = \|\vec{k}_{out} - \vec{k}_{in}\|$ is the momentum transfer.

(b) *Evaluate the scattering amplitude for the potential*

$$V(r) = \begin{cases} V_0\frac{1}{r} & r \le a \\ 0 & r > a \end{cases} \qquad r = |\vec{r}|, \quad a > 0.$$