(1) Contextuality inequalities **3 points**

In the video lecture "Contextuality" you heard how the perfect anticorrelations obtained by looking through two out of three holes using binoculars are incompatible with the global assignment of all the values to three holes. To say it otherwise, it is impossible to complete the table for probabilistic outcomes in a way that the values $\pm$ are assigned to all three holes such that every pair is perfectly uncorrelated. To step back and look at a bigger picture, these findings let us make a very important conclusion: *Sometimes there are datasets incompatible with the assumption that the physical properties pre-exist and measurements are there to only discover them!*

To formalise the discussion above, the correlators $S_i S_{i+1}$ were defined, where the addition in the subscript is *cyclic* (for three holes this means $i+1 \mod 3$) and $S_i \in \{+1,-1\}$. Using this notation and the experiment described above, in the lecture the linear inequalities where obtained for three observed variable:

$$\frac{1}{3}\sum_{i}^{3}\mathbb{E}[^3 S_i^{(k)} S_{i+1}^{(k)}] \geq -\frac{1}{3}. \tag{1}$$

Now we shall generalise this inequality to N observed variables, which are all arranged in a circle. We again consider the nearest-neighbour correlators only and the cyclicity implies that $N+1$-th variable corresponds to the first one.

(a) Derive the 5-variable generalisation of the linear contextual inequality, thus generalise Eq. (1) for five variables.

(b) Now take an odd N. Derive the N-variable generalisation of the linear contextual inequality. Thus generalise Eq. (1) for odd N variables.

(c) Now take an even N. Does the direct generalisation work in this case and why?

(2) Statistics of CHSH experiments **7 points**

Recall CHSH inequality and the corresponding Bell test described in the video lecture "CHSH" and "LHV models". Denote by $(M_1^{(i)}, M_2^{(i)}, N_1^{(i)}, N_2^{(i)})$ a vector of four numbers in $\{\pm 1\}$ for $i = 1, \ldots, n$, which can be interpreted as the *partly unmeasured* outcomes of the four possible measurements performed in the i-th run of a CHSH experiment. The goal of this exercise is to get a feeling how well the quantity $C = M_1N_1 + M_1N_2 + M_2N_1 - M_2N_2$ can be estimated if we have access only to a randomly chosen pair $(M_{X_i}^{(i)}, N_{Y_i}^{(i)})$ of measurements per run. Here, we assume that X_i and Y_i are independent true random variables that take on the values $\{1, 2\}$.

We'll make use of the *Chernoff-Hoefding inequality* (a proof of which you are encouraged to look up). It says that if $A_1, \ldots, A_n$ are independent random variables that take values in $[-1, 1]$, and

$$S_n := \frac{1}{n}\sum_{i=1}^{n} A_i$$

is their mean, then, for all $t > 0$,

$$\Pr\Big[\big|S_n - \langle S_n\rangle\big| \geq t\Big] \leq 2e^{-\frac{nt^2}{2}}.$$

Here, $\langle S_n \rangle$ is the expectation value of S_n. In other words, the bound says that the probability that S_n will deviate from its expected value is exponentially small in the squared deviation.

(a) Focus on the first summand in C. Set

$$A_i = \begin{cases} 4M_1^{(i)} N_1^{(i)} & \text{if } X^{(i)} = Y^{(i)} = 1 \\ 0 & \text{else} \end{cases}$$

Show that $\langle S_n \rangle = \frac{1}{n} \sum_{i=1}^{n} M_1^{(i)} N_1^{(i)}$. This justifies the use of S_n as an estimate for the mean value of $M_1^{(i)} N_1^{(i)}$ over the table. The other summands can be estimated similarly.

(b) Now assume that the estimation procedure above has been performed for $n = 1000$ runs and resulted in an estimate of 2.8 for the mean of C over the table. Of course, it could be that the true mean value of C is actually smaller than or equal to 2 and that the apparent larger value is a statistical fluke. Prove that this is less likely than winning the "6 out of 49" lottery. Hint: Use that at least one of the four estimated summands must deviate by at least .2 from its actual mean. (4)