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(3) Clifford algebras and their properties **5 points**

In the video "CHSH violations" we discussed Clifford algebras and their properties. Here we will prove these properties. Recall that matrices, γ_i, $i \in 1, \dots n$ which satisfy the condition in Eq. (1) are generators of a Clifford algebra:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}\mathbb{1} \,. \tag{1}$$

Now let $\gamma_1, \dots, \gamma_n$ be $d \times d$ matrices satisfying the condition in Eq. (1) and for $\vec{a} \in \mathbb{R}$ let us define an operator as follows:

$$\gamma(\vec{a}) = \sum_i^n a_i \gamma_i. \tag{2}$$

Prove the following three properties:

(a) $\gamma(\vec{a})^2 = ||\vec{a}||^2 \cdot \mathbb{1}_d$. **1 point**

(b) Show that $\gamma(\vec{a})$ has eigenvalues $\pm||\vec{a}||$ with equal multiplicity. **2 points**

(b) $\mathrm{Tr}\left[\gamma(\vec{a}) \cdot \gamma(\vec{b})\right] = d \cdot (\vec{a}, \vec{b})$. **2 points**

(4) Bell inequalities without inequalities **7 points**

(a) In the spirit of the first lecture-set show that the following expressions, where all the variables (classical) have the values ± 1 cannot be satisfied simultaneously: **2 points**

$$\begin{aligned} X_1 X_2 X_3 &= 1 \\ -X_1 Y_2 Y_3 &= 1 \\ -Y_1 X_2 Y_3 &= 1 \\ -Y_1 Y_2 X_3 &= 1. \end{aligned}$$

Now consider a quantum state $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, where $|000\rangle \equiv |0\rangle \otimes |0\rangle \otimes |0\rangle$. If you do not know what $\otimes$ means, please, look up *tensor product* or ask in the exercise class. In this exercise X_i, Z_i and Y_i are operators (in the last one they were just numbers) and correspond to Pauli matrices.

(b) Show that the state $|\psi\rangle$ is the eigenstate of the following operators with the eigenvalue "+1". Thus, prove the following equations: **1 point**

$$\begin{aligned} X_1 \otimes X_2 \otimes X_3 |\psi\rangle &= |\psi\rangle \\ \mathbb{1}_1 \otimes Z_2 \otimes Z_3 |\psi\rangle &= |\psi\rangle \\ Z_1 \otimes \mathbb{1}_2 \otimes Z_3 |\psi\rangle &= |\psi\rangle \\ Z_1 \otimes Z_2 \otimes \mathbb{1}_3 |\psi\rangle &= |\psi\rangle. \end{aligned}$$

(c) The operators in the part (b) are generators of so-called stabiliser group. By multiplying some of them show that we can get the following equations: **1 point**

$$-X_1 \otimes Y_2 \otimes Y_3|\psi\rangle = |\psi\rangle$$
$$-Y_1 \otimes X_2 \otimes Y_3|\psi\rangle = |\psi\rangle$$
$$-Y_1 \otimes Y_2 \otimes X_3|\psi\rangle = |\psi\rangle.$$

(d) Interpret the results of the parts (a) and (c). **2 point**