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(8) Extremal quantum mechanical states **50 points**

In the lecture we shortly touched the subject of extremal points of quantum state space. We also know that reversible dynamics of QM are given by Hamiltonian time evolutions and these give rise to unitary transformations:

$$U|\psi(0)\rangle = |\psi(t)\rangle, \quad \text{where} \quad U = e^{-itH}. \tag{1}$$

In this exercise we stick to states of at most two qubits, we transform them using unitary evolution and check if they give raise no noncontextual HVM.

(a) Show that an arbitrary single qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ can be brought to $|0\rangle$ using some single-qubit unitary action on $|\psi\rangle$. **5 points**

$$U|\psi\rangle \mapsto |0\rangle. \tag{2}$$

(b) Consider an arbitrary two qubit pure state $|\phi\rangle = \sum_{i,j} c_{ij}|ij\rangle$. Show that the state $|\phi\rangle$ can be prepared from a two qubit product state $|00\rangle$ using tensor product of two unitary operations $U_1 \otimes U_2$, when $|\phi\rangle$ is not entangled. U_i acts on a qubit i. **5 points**

(c) Prove that an arbitrary two qubit pure state $|\phi\rangle$ can be written as follows for some unitaries $U_1' \otimes U_2'$: **15 points**

$$U_1' \otimes U_2'|\phi\rangle = |\eta\rangle = \sin\alpha|00\rangle + \cos\alpha|11\rangle. \tag{3}$$

This form of expressing a state is called Schmidt decomposition (You cannot use this decomposition in the solution (b)).

(d) Recall the Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. That is $\alpha = \frac{\pi}{4}$ in Eq. (3). Show that this state can be prepared by action of two qubit unitary operation U_{12} on a product state $|00\rangle$, where **5 points**

$$U_{12} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{4}$$

(e) We saw in the lecture that a Bell state violates CHSH inequality: **20 points**

$$\langle \mathcal{B} \rangle = \langle A_1 \otimes B_1 \rangle + \langle A_1 \otimes B_2 \rangle + \langle A_2 \otimes B_1 \rangle - \langle A_2 \otimes B_2 \rangle \leq 2. \tag{5}$$

In this exercise you have to show that for any two qubit entangled state $|\eta\rangle$, there exist some observables A_i and B_j such that

$$\langle \mathcal{B} \rangle_{|\eta\rangle} > 2. \tag{6}$$

Hint: To do this, first fix $A_1 = \sigma_z$ and $A_2 = \sigma_x$. Take $B_1 = \cos b_1\sigma_x + \sin b_1\sigma_z$ and $B_2 = \cos b_2\sigma_x + \sin b_2\sigma_z$. Insert all these values in $\langle \mathcal{B} \rangle_{|\eta\rangle}$ and optimize over b_1 and b_2 for arbitrary α.