Lecturer: David Gross. Exercises: Mariami Gachechiladze, Lukas Franken.
E-mail for submissions/questions: marigachi@gmail.com

(10) SIC-POVMs — 40+∞ points

In the last lectures you learnt that the types of measurements we used to consider until now were not the most general ones. Accordingly, now that we know the truth, we have to explore it. Thus in this sheet we will work on POVMs. A POVM over a d-dimensional Hilbert space $\mathcal{H}$ is a set of n positive-semidefinite operators $E = \{E_1, E_2, \dots E_n\}$ on the Hilbert space that sum to the identity: $\sum_{i=1}^{n} E_i = \mathbb{1}_d$.

In the lecture you briefly heard about symmetric informationally complete positive operator-valued measure (SIC-POVM), which is a special case of a generalised measurement. If a POVM consists exactly of d^2 rank-1 operators, $E = \{E_1, E_2, \dots E_n\}$, which can be defined using a set of projectors, $\Pi = \{\Pi_1, \Pi_2, \dots \Pi_n\}$ in the following way: $E_i = \Pi_i/d$, and if these rank-1 projectors satisfy a pair-wise relation,

$$\mathrm{Tr}(\Pi_i \cdot \Pi_j) = \frac{d\delta_{ij} + 1}{d+1}, \quad \text{for all} \;\; 1 \leq i, j \leq n, \tag{1}$$

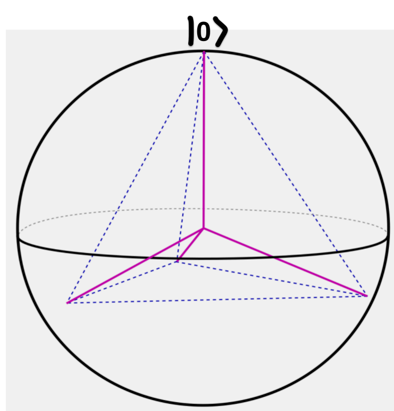
then POVM is said to be a symmetric informationally complete POVM (SIC-POVM).

(a) Find a set Π for $d = 2$. See the figure for the possible visualisation of the projectors on a Bloch sphere. You can start by fixing $\Pi_1 = |0\rangle\langle 0|$.

(b) Fill in the remaining labels on the figure for all the projectors you find (If the projectors look different, feel free to redraw the figure)

(c) Is the choice unique even if $\Pi_1 = |0\rangle\langle 0|$ is fixed?

(d) Show that the renormalised projectors to the rank-1 effects E_i indeed form a POVM.



($\star$) Bonus: Do SIC-POVMs exist in all dimensions?

(11) Unambiguous state discrimination — 10 points

Suppose Alice and Bob are playing a game, where Alice prepares linearly independent state set $S = \{|\psi_1\rangle, |\psi_2\rangle \dots |\psi_n\rangle\}$ and then sends one of these n states $|\psi_i\rangle$ to Bob. Bob is aware of states in S, but he does not know which state has Alice sent. You shall help Bob to construct a POVM $\{E_1, E_2, \dots E_{n+1}\}$, such that when the outcome E_i occurs, Bob can guess with certainty that Alice sent $|\psi_i\rangle$.