Winter Semester 2022/23

Foundations of Quantum Mechanics

Homework Sheet 5

Lecturer: David Gross
Exercises: Vahideh Eshaghian

- To be submitted to veshaghi@uni-koeln.de by January 15th.

1. **SIC-POVMs**

In this sheet, we will have a look at a particularly elegant class of quantum measurements, with a particularly inelegant name: *symmetric informationally complete positive operator-valued measures*. Or SIC-POVMs. Or just SICs. (Pronounced *seeks*, because SIC sounds sick, and because their general existence isn't yet conclusively proven, so people are still, well, *seeking* them). (Yeah, I, too, wish I made this up).

Anyway, recall that as we saw in the lecture, each quantum measurement with k outcomes is associated with a collection of positive semi-definite operators, $\{E_i\}_{i=1}^k$, whose sum equals the identity operator.

A collection of normalized vectors $\{|\phi_i\rangle\}_{i=1}^{d^2} \subset \mathbb{C}^d$ is a *SIC* if it satisfies the following conditions:

i. The associated rank-1 projections, appropriately weighted, sum to $\mathbb{1}$:

$$\sum_i \frac{1}{d} |\phi_i\rangle\langle\phi_i| = \mathbb{1}.$$

In other words, the set of $E_i = \frac{1}{d}|\phi_i\rangle\langle\phi_i|$ forms a POVM.

ii. Their mutual inner product equals some constant $c \in \mathbb{R}$:

$$|\langle\phi_i|\phi_j\rangle|^2 = c, \qquad \forall i \neq j.$$

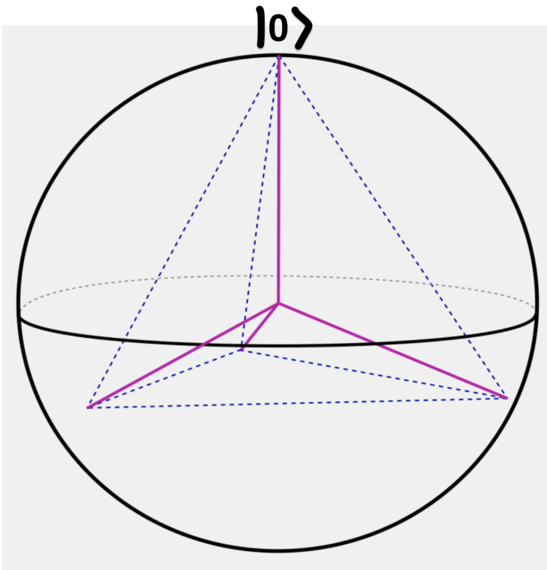
It is in this sense that the elements form a *symmetric* configuration.

Let's work out some properties:

(a) *Prove that these conditions force $c = 1/(d+1)$.* Hint: Use that $(A|B) := \operatorname{Tr} A^\dagger B$ defines a scalar product on operators and evaluate the condition $\| \sum_i E_i - \mathbb{1}\|^2 = 0$. (5 points)

(b) Next, we prove that ρ can be reconstructed from the probabilities $p_i = \operatorname{Tr} \rho E_i$ for $i = 1, \dots, d^2$. This is what *informational completeness* refers to, and which distinguishes such POVMs from the usual basis measurements which always loose phase information. *Show that there is a constant λ such that, with $F_i = E_i + \lambda\mathbb{1}$, the operators F_i become orthogonal to each other with respect to the scalar product introduced above. Argue why that means that ρ can be reconstructed from the inner products $(\rho|F_i)$. Argue that the $(\rho|F_i)$ are functions of the probabilities $\operatorname{Tr} \rho E_i$.* Hint: This is basic linear algebra. Recall the dimension of the space of $d \times d$-matrices. (5 points)

(c) For $d = 2$, the squared inner product between pure states is a function of the inner product of their vectors on the Bloch sphere. Thus, four equiangular Bloch vectors give rise to a POVM. *Construct a SIC from the edges of a tetrahedron inscribed into the Bloch sphere.*



(10 points)

(⋆) Bonus: Do SIC-POVMs exist in all dimensions? (This is a joke. Don't work on this problem!) (∞ points)