Winter Semester 2022/23

Foundations of Quantum Mechanics

Homework 1

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Exercises: Vahideh Eshaghian

- To be submitted to veshaghi@uni-koeln.de by 23rd October.
- Submissions in groups of three people are strongly encouraged!

1. **Contextuality inequalities** (3 Points)

As in the first lectures, consider three variables S_1, S_2, S_3 that take values in $\{+1, -1\}$. We will be analyzing *correlation functions* between neighboring variables, i.e. expectation values of the form $\mathbb{E}[S_i S_{i+1}]$. Here, the indices in the subscript are *cyclic* (for three properties this means that the indices are to be read modulo 3, so e.g. $S_4 = S_1$). Using this notation, the contextuality inequality obtained in the lecture reads:

$$\frac{1}{3}\sum_{i=1}^{3}\mathbb{E}[S_i S_{i+1}] \geq -\frac{1}{3}.$$

Now the goal is to generalise this inequality to K observed variables. In other words, we aim to find the largest number C_K such that

$$\frac{1}{K}\sum_{i=1}^{K}\mathbb{E}[S_i S_{i+1}] \geq C_K \tag{1}$$

is valid for all non-contextual models.

(a) Find C_5. (This is the most important value, as we will later see that 5 is the smallest value for K such that the inequality is known to be violated by physical experiments).

(b) Find C_K for general *odd* $K \geq 3$.

(c) Find C_K for general *even* K. Does it make sense to consider this case? Justify your answer.

2. **Statistics of CHSH experiments** (7 Points)

Recall the CHSH inequality described in the lecture. Assume (for the sake of reaching a contradiction) that there is a non-contextual assignment of values $\{\pm 1\}$ to the properties $(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)})$. The superscript $i = 1, \ldots, n$ refers to the run of the experiment.

The goal of this exercise is to get a feeling of how well the quantity,

$$C := \mathbb{E}[A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2] = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(A_1^{(i)}B_1^{(i)} + A_1^{(i)}B_2^{(i)} + A_2^{(i)}B_1^{(i)} - A_2^{(i)}B_2^{(i)}\right)$$

can be estimated if we have access only to a randomly chosen pair $(A_{X_i}^{(i)}, B_{Y_i}^{(i)})$ of measurements per run. Here, we assume that X_i and Y_i (i.e. the choices of Alice and Bob of which property to measure in the i-th run) are independent random variables that take on the values $\{1,2\}$ with probability $1/2$ each.

We'll make use of the *Chernoff-Hoefding inequality* (a proof of which you are encouraged to look up). It says that if $M_1, ..., M_n$ are independent random variables that take values in $[-1,1]$, and

$$S_n := \frac{1}{n}\sum_{i=1}^{n} M_i,$$

is their mean, then, for all $t > 0$,

$$\Pr\left[|S_n - \mathbb{E}[S_n]| \geq t\right] \leq 2e^{-\frac{nt^2}{2}}.$$

In other words, the bound says that the probability that S_n will deviate from its expected value is exponentially small in the squared deviation.

(a) Focus on the first summand in C. Set

$$M_i = \begin{cases} 4A_1^{(i)}B_1^{(i)} & \text{if } X^{(i)} = Y^{(i)} = 1 \\ 0 & \text{else} \end{cases}$$

Show that $\mathbb{E}[S_n] = \frac{1}{n}\sum_{i=1}^{n} A_1^{(i)}B_1^{(i)}$. This justifies the use of S_n as an estimate for the mean value of $A_1^{(i)}B_1^{(i)}$ over the table. The other summands can be estimated similarly.

(b) Now assume that the estimation procedure above has been performed for $n = 1000$ runs and resulted in an estimate of 2.8 for the mean of C over the table. Of course, it could be that the true mean value of C is actually smaller than or equal to 2 and that the apparent larger value is a statistical fluke. Prove that this is less likely than winning the "6 out of 49" lottery.
Hint: Use that at least one of the four estimated summands must deviate by at least 0.2 from its actual mean.