Winter Semester 2022/23

Foundations of Quantum Mechanics

Homework 2

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- To be submitted to veshaghi@uni-koeln.de by 13th November.

Note: There's a lot of text on this sheet! But most of it is background information – there's not actually that much to compute.

1. **Bell inequalities without inequalities** (7 Points)

Recall that, while deriving the CHSH inequality, we found that classical variables taking values in ± 1 cannot simultaneously satisfy the conditions

$$A_1B_1 = 1, \quad A_1B_2 = 1, \quad A_2B_1 = 1, \quad A_2B_2 = -1.$$

But (as we will prove later), these relations are also impossible to realize in quantum mechanics (a Popescu-Rohrlich box could do it, but alas, none has ever been observed). That's why we had to resort to a more subtle and quantitative criterion given by the CHSH inequality.

On this sheet, we will show that such deterministic violations *are* possible in QM after all – at least if one involves three parties!

This phenomenon was discovered by Greenberger-Horn-Zeilinger (GHZ). The "Z" of GHZ and the "C" of CHSH were, of course, awarded the 2022 Nobel Prize for their work on Bell inequalities!

(a) Consider six classical variables $X_1, X_2, X_3, Y_1, Y_2, Y_3$ taking values in ± 1. Show that the following four equations cannot be satisfied simultaneously. **(2 points)**

$$\begin{aligned} X_1X_2X_3 &= 1, \\ -X_1Y_2Y_3 &= 1, \\ -Y_1X_2Y_3 &= 1, \\ -Y_1Y_2X_3 &= 1. \end{aligned} \tag{1}$$

We will now show that there are quantum-mechanical ways of measuring the X_i's and Y_i's such that each of the equations in (1) holds deterministically! (Just to be sure: We cannot measure X_i and Y_i *simultaneously* – so a contradiction only arises if we counter-factually assume that they possess values independently of our choices of which ones to observe).

We need a minimum of QM, which we recap now.

Consider a single-particle Hilbert space spanned by two ortho-normal states $\{|0\rangle, |1\rangle\}$ (e.g. the "spin-up" and "spin-down" states with respect to the z-axis, of a spin-1/2 particle). The *Pauli matrices* $\{\sigma_x, \sigma_y, \sigma_z\}$ are defined via their action on this basis:

$$\begin{aligned} \sigma_x|0\rangle &= |1\rangle, & \sigma_x|1\rangle &= |0\rangle, \\ \sigma_y|0\rangle &= i|1\rangle, & \sigma_y|1\rangle &= -i|0\rangle, \\ \sigma_z|0\rangle &= |0\rangle, & \sigma_z|1\rangle &= -|1\rangle. \end{aligned}$$

Below, we will not work with a single, but with three particles! We use superscripts to indicate which particle an operator acts on. For example:

$$\sigma_x^{(2)}|0,0,0\rangle = |0,1,0\rangle, \qquad \sigma_z^{(1)}|1,0,0\rangle = -|1,0,0\rangle.$$

(b) The *GHZ state* is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Prove the following eigenvalue equations: **(3 point)**

$$\begin{aligned}
\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}|\psi\rangle &= |\psi\rangle \\
-\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}|\psi\rangle &= |\psi\rangle \\
-\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}|\psi\rangle &= |\psi\rangle \\
-\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}|\psi\rangle &= |\psi\rangle.
\end{aligned}$$

Hint: You can verify these equations directly using the definition of the Pauli matrices. If you remember their composition laws from your undergrad class (or if you care to look them up on Wikipedia), there's a slightly more elegant way. Indeed, argue that the following equations can be seen to hold, without performing any calculations:

$$\begin{aligned}
\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}|\psi\rangle &= |\psi\rangle \\
\mathbb{1}^{(1)}\sigma_z^{(2)}\sigma_z^{(3)}|\psi\rangle &= |\psi\rangle \\
\sigma_z^{(1)}\mathbb{1}^{(2)}\sigma_z^{(3)}|\psi\rangle &= |\psi\rangle \\
\sigma_z^{(1)}\sigma_z^{(2)}\mathbb{1}^{(3)}|\psi\rangle &= |\psi\rangle.
\end{aligned}$$

Now show that the product of the operator in the first equation and the one in the second equation gives you one of the operators that occur in the original set of eigenvalue equations. Go on from here.

(c) Recall that the Pauli matrices are Hermitian operators with eigenvalues ± 1. Interpret the results! What do these calculations have to do with the claim made after part 1(a)? **(2 points)**

Side note:
There is a nice popular science article by Nobel laureate Frank Wilczek, which talks about the contradiction in Exercise 1(a). Indeed, search for the paragraph that starts with "But later" in `https://www.quantamagazine.org/entanglement-made-simple-20160428`. One of us (DG) is rather confused about this paragraph and the one that follows. The author notes that there is an *odd* number of +1's (called "the evil outcome" in this article) among the properties measured in the first case, but an *even* number among the properties measured in the other cases. The article then seems to imply that this constitutes an obvious contradiction ("So: Is the quantity of evil even or odd?").

But things are just a little bit more complicated! The fact that the number of +1's observed changes depending on which quantities are observed is not very surprising and does not, by itself, imply contextuality. Indeed: I invite you to write down a assignment that satisfies the first two (or even the first three) equations above. In the end, there *is* of course a contradiction! But it depends on the specific structure of the four triples of observations.

Now the dude does have a Nobel, so maybe I'm the one who's confused? There's extra points to whoever finds a mistake with my reasoning.

Bonus exercise: In the spirit of Wilczek, show that (1) is inconsistent by proving that some lines imply that the number of -1's *among the variables* X_1, X_2, X_3 *alone* is even, while some imply that this number is odd.