Winter Semester 2022/23

Foundations of Quantum Mechanics

Homework Sheet 3

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Exercises: Vahideh Eshaghian

- To be submitted to veshaghi@uni-koeln.de by November 27th.

1. **All the contextuality inequalities (for three correlators)**

   We started this lecture by considering a contextuality inequality for the two-point correlation functions $\mathbb{E}[S_1S_2], \mathbb{E}[S_2S_3], \mathbb{E}[S_3S_1]$ of three binary variables $S_1, S_2, S_3 \in \{\pm 1\}$. Let's call this the *monolith scenario*.

   For the more complicated CHSH scenario (a set of four two-point correlation functions involving the four binary variables $A_1, A_2, B_1, B_2 \in \{\pm 1\}$, we later used a computer algebra system to find *all* contextuality inequalities. The CHSH inequality was one of them, and all other ones resulted from the symmetry operations $A \leftrightarrow B$ and $1 \leftrightarrow 2$.

   In this exercise, you are asked to complete a similar analysis for the (simpler!) monolith scenario.

   (a) An analogy to the CHSH scenario discussed in the lecture, every joint probability distribution of S_1, S_2, S_3 gives rise to a vector $(\mathbb{E}[S_1S_2], \mathbb{E}[S_2S_3], \mathbb{E}[S_3S_1]) \in \mathbb{R}^3$. The set of all these vectors is a convex polytope whose vertices correspond to deterministic assignments. Find these vertices. (2 points)

   (b) Find the linear inequalities corresponding to the faces of the polytope. (4 points)
   Hint: As the polytope lives in three dimensions, one can conceivably find the inequalities with pen, paper, and eyeballs. A computer algebra system is also acceptable. In the lecture, we used `SageMath`. You can install it on your computer for free, or access the cloud version at `https://sagecell.sagemath.org/`. Click here for some relevant documentation.

   (c) In addition to the one inequality derived in the lecture, you should have found three more. Show that, as was the case in the CHSH scenario, these too all arise from each other from natural symmetry transformations of three binary variables. (4 points)

   (d) Bonus: The polytope you've encountered is a real classic! What's its name?