Winter Semester 2022/23

Foundations of Quantum Mechanics

Homework Sheet 4

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1. **State-independent contextuality**

We have encountered three examples of contextual distributions that can be realized using quantum mechanical systems. In all these cases (CHSH, GHZ, and Klyachko's pentagram), we had to carefully design both the observables and the quantum state such as to violate the associated contextuality inequality. Given this experience, it may come as a surprise that there are settings that show *state-independent contextuality*: These are observables that give contextual results when measured in *any* quantum state. Here, we explain the best-known construction, the *Mermin-Peres square.*

Consider a 3×3-square, where each cell is associated with a ± 1-valued variable. Let's describe an assignment by values $S_{ij} \in \{\pm 1\}, i, j = 1, \ldots, 3$, where i indexes the row and j indexes the column. We will assume that the variables in each row and in each column are jointly observable.

(a) *Show that there is no assignment such that the product of the variables along each of the three rows equals* $+1$*, whereas the product of the variables along each of the three columns equals* -1. Formally, the following system of equations are not satisfiable: (2 points)

$$\prod_j S_{ij} = +1 \quad (i = 1 \ldots 3), \qquad \prod_i S_{ij} = -1 \quad (j = 1 \ldots 3). \tag{1}$$

We will now construct a realization on two qubits. To this end, assign quantum observables to the variables according to the following scheme:

$\hat{S}_{1,1} = \sigma_z^{(1)}$	$\hat{S}_{1,2} = \sigma_z^{(2)}$	$\hat{S}_{1,3} = \sigma_z^{(1)} \sigma_z^{(2)}$
$\hat{S}_{2,1} = \sigma_x^{(2)}$	$\hat{S}_{2,2} = \sigma_x^{(1)}$	$\hat{S}_{2,3} = \sigma_x^{(1)} \sigma_x^{(2)}$
$\hat{S}_{3,1} = -\sigma_z^{(1)} \sigma_x^{(2)}$	$\hat{S}_{3,2} = -\sigma_x^{(1)} \sigma_z^{(2)}$	$\hat{S}_{3,3} = \sigma_y^{(1)} \sigma_y^{(2)}$

(b) We'll start by focusing on the first row. *Show that the three observables in the first row mutually commute, and that their product is the identity operator.* As we will work out next, this means that these three observables are jointly measurable and that the product of the outcomes is always $+1$. Indeed, recall from your undergrad QM class that mutually commuting operators have a joint eigenbasis. *Find the joint eigenbasis. For each of the four eigenvectors, give the value of* $\hat{S}_{1,1}, \hat{S}_{1,2}, \hat{S}_{1,3}$ *and show that their product is* $+1$.

(4 points)

(c) *Show that the operators in* each *of the rows and each of the columns mutually commute. Further show that their product along every row equals the identity, and that the product along every column equals minus the identity.* (5 points)

(d) Now let $|\psi\rangle$ be an *arbitrary* two-qubit state vector. *Conclude from the above that*

$$\prod_j \langle\psi|\hat{S}_{ij}|\psi\rangle = +1 \quad (i = 1 \ldots 3), \qquad \prod_i \langle\psi|S_{ij}|\psi\rangle = -1 \quad (j = 1 \ldots 3).$$

By comparison with (1), it follows that the values assigned to rows / columns of the square by the quantum model are incompatible with any full classical assignment, i.e. they are contextual. (2 points)

(e) [Bonus exercise] We can express this conclusion equivalently via a contextuality inequality. *To this end, use the first result to argue that for classical joint distribution,*

$$\sum_{i=1}^{3} \mathbb{E}\Big[\prod_{j} S_{ij}\Big] - \sum_{j=1}^{3} \mathbb{E}\Big[\prod_{i} S_{ij}\Big] \leq 4,$$

while the quantum realization achieves a value of 6.