Winter Semester 2022/23

Foundations of Quantum Mechanics

Homework Sheet 6

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Exercises: Vahideh Eshaghian

- This is a bonus exercise! You can submit it if you wish to collect more points.
- To be submitted to veshaghi@uni-koeln.de by January 29th.

1. **Why do we see a classical world?**

The answer lies in *decoherence*, the omnipresent and (almost) inevitable interaction with the environment, which causes superpositions on macroscopic scales to disappear. We now see how. As a simple model, we consider a macroscopic dust grain with spatial degree of freedom $|x\rangle$, that scatters a photon in the coherent state $|L\rangle$. Recall that coherent state are the eigenstates of the annihilation operator of a harmonic oscillator, $a|L\rangle = L|L\rangle$, $L \in \mathbb{C}$.

The dust grain remains unchanged, while the photon undergoes a displacement of $x\sqrt{2\Lambda}$. Displacement operator is defined as follows:

$$D(l) \coloneqq e^{l(a^\dagger - a)}, \tag{1}$$

here $a^\dagger, a$ are the creation and annihilation operators, respectively. So, the scattering from position x is given by $|x\rangle|L\rangle \mapsto |x\rangle D(x\sqrt{2\Lambda})|L\rangle$.

(a) Show that $D(l_1)D(l_2) = D(l_1 + l_2)$. (1 point)

(b) Assume that the dust grain is initially in a superposition of two positions x_1 and x_2, in the state

$$|\psi\rangle = \alpha|x_1\rangle + \beta|x_2\rangle.$$

Show that the state of the dust grain and the photon after the scattering is

$$|\Psi\rangle = \alpha|x_1\rangle|L + x_1\sqrt{2\Lambda}\rangle + \beta|x_2\rangle|L + x_2\sqrt{2\Lambda}\rangle.$$

Hint: $|L\rangle = D(L)|0\rangle$, for coherent states $|L\rangle$ and $|0\rangle$. (1 point)

(c) Recall the partial trace of a state from the lecture. For a two-particle system in state $|\gamma\rangle = \sum_{i,j} \alpha_{ij}|\phi_i\rangle|\phi_j\rangle$, the partial trace over the second particle is defined as:

$$\begin{aligned}\mathrm{Tr}_2(|\gamma\rangle\langle\gamma|) &= \mathrm{Tr}_2\Big(\sum_{ijkl} \alpha_{ij}\bar{\alpha}_{kl}|\phi_i\rangle\langle\phi_k| \otimes |\phi_j\rangle\langle\phi_l|\Big)\\ &= \sum_{ijkl} \alpha_{ij}\bar{\alpha}_{kl}|\phi_i\rangle\langle\phi_k|\langle\phi_l|\phi_j\rangle\end{aligned}$$

Compute the four matrix elements of the reduced density operator between the states $\{|x_1\rangle, |x_2\rangle\}$.
Describe what's happening. (4 points)

(d) To generalize the model, we now assume that the dust grain is initially described by an arbitrary wave function $\phi(x)$ (instead of by a superposition of only two locations). The scattered state is then

$$\int dx \phi(x)|x\rangle|L\rangle \longrightarrow \int dx \phi(x)|x\rangle D(x\sqrt{2\Lambda})|L\rangle,$$

and the density operators of the dust grain is

$$\begin{aligned}\rho &= \mathrm{Tr}_2 \int dx dx' \phi(x)\bar{\phi}(x')|x\rangle\langle x'| \otimes |L + x\sqrt{2\Lambda}\rangle\langle L + x'\sqrt{2\Lambda}| \\ &= \int dx dx' \phi(x)\bar{\phi}(x')|x\rangle\langle x'|\langle L + x'\sqrt{2\Lambda}|L + x\sqrt{2\Lambda}\rangle.\end{aligned}$$

Compute the matrix elements

$$\langle x|\rho|x'\rangle \tag{2}$$

of the reduced density matrix. (4 points)

We'll look at realistic rates for macroscopic objects during the exercise.